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Estimation of moments of a Poisson-sampled random process

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Abstract. Unbiased consistent estimators of the mean and variance of a correlated sequence $\{x(t_k)\}$, $k = 1, 2, \ldots, n$, are derived. The sequence is obtained by Poisson sampling a random process $x(t)$ with a known covariance function. The variances of the estimators are obtained. Finally, the application of the results to laser anemometry is discussed.

1. Introduction

There has been a considerable interest in the statistics of point processes. Beutler and Leneman (1966) have obtained expressions for the spectrum of a Poisson-sampled process. However, they did not consider the sampling distribution of the spectrum.

In this paper, the unbiased mean and variance estimators of a correlated sequence $\{x(t_k)\}$, $k = 1, 2, \ldots, n$, are derived. The continuous process $x(t)$ is assumed to have a known covariance function (CF) and the sampling to be Poisson, i.e. the number of samples in a fixed time is Poisson distributed with rate ν. Three specific CFs are considered—$\exp(-\alpha|\tau|)$, $\exp(-\alpha|\tau|)\cos(\omega\tau)$ and $\exp(-\lambda\tau^2)$.

The variances of the mean estimators are obtained without any assumptions regarding the statistics of $x(t)$. In order to evaluate the variance of the variance estimators, Gaussian statistics for $x(t)$ have been assumed. The results for the CF $\exp(-\alpha|\tau|)\cos(\omega\tau)$ can be obtained directly from the exponentially decaying CF $\exp(-\alpha|\tau|)$. Hence, only the CF $\exp(-\alpha|\tau|)$ is discussed in detail. Formulae are obtained for the limiting cases when the number of samples n is large. Although the form of the variances of the estimators for the Gaussian CF $\exp(-\lambda\tau^2)$ are similar to those for the CF $\exp(-\alpha|\tau|)$, the results cannot be derived analytically, except in the limiting case of large n for the mean value estimator. With the aid of a recurrence formula, however, numerical results can easily be obtained.

A possible area of application of these results is in laser anemometry which is a widely used non-contact method of fluid flow measurement. In this technique laser light is focused into the flow region and a signal is obtained from light scattered by minute particles suspended in the fluid. These particles are either naturally present or

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artificially seeded and are generally so small that they can be assumed to follow faithfully the fluid motions. The scattered light is observed with a photodetector and the resulting electronic output is converted to a velocity record by means of an electronic processing system. Full details of these techniques are given by Durrani and Greated (1977). The random-sampling problem arises since in many practical situations, particularly in the measurement of air flows, the particles are widely dispersed within the fluid. In this case the photodetector signal and the resulting velocity record is in the form of a discrete series of pulses, each pulse arising when a particle passes across the measuring region.

In contrast to laser anemometry the more conventional measuring technique known as hot-wire anemometry gives rise to a continuous velocity record and hence the random-sampling problem does not arise. It is worth pointing out, however, that the laser methods have a number of fundamental advantages over hot-wire anemometry. These are fully described in the literature (Durrani and Greated 1977). In § 6 we have compared the errors arising in random sampling with those obtained when a continuous record is employed. It is intuitively obvious that for a given sample time the errors are greater with a random sample and that the errors become equivalent in the limiting case of infinitely high sample rate.

The scattering particles in laser anemometry are randomly dispersed within the fluid. Hence if a vanishingly small sample volume is chosen within the flow region, the probability of finding a scattering centre within this volume at a given instant will be a constant, independent of the spatial position, and this probability will also be vanishingly small. This is the spatial equivalent to the shot noise process. Our results are applicable to flows where there are small random velocity fluctuations superimposed on a constant mean velocity. In this situation it is seen that the temporal distribution of point velocity readings, as well as the spatial distribution of particles within the flow region, will be Poisson to a good approximation. If velocity fluctuations are excessively large then bunching effects occur (McLaughlin and Tiederman 1973).

2. Poisson sampling

If the number of samples in a fixed time has a Poisson distribution, then the interval (τ) between adjacent samples has an exponential probability density and the probability density of any two non-overlapping intervals is independent. The probability density of τ is

$$p(\tau)=\begin{cases}\nu\exp(-\nu\tau) & \tau\geqslant 0\\ 0 & \tau<0\end{cases} \tag{2.1}$$

where ν is the rate parameter of the Poisson distribution. From this it follows that the probability density of the interval between any two samples $x(t_m)$ and $x(t_{m+n})$ separated by n independent intervals is

$$\begin{aligned} p_n(\tau)&=\int_{-\infty}^{+\infty} p(u)p_{n-1}(\tau-u)\,\mathrm{d}u=\int_0^{\tau} p(u)p_{n-1}(\tau-u)\,\mathrm{d}u\\ p_1(\tau)&\equiv p(\tau)\end{aligned} \tag{2.2}$$

i.e. $p_n(\tau)$, $n\geqslant 2$, is obtained by successively convolving $p(\tau)$ with itself $(n-1)$ times. This leads to

$$p_n(\tau)=\begin{cases}\dfrac{\nu^n\tau^{n-1}}{(n-1)!}\exp(-\nu\tau) & \tau\geqslant 0\\ 0 & \tau<0.\end{cases} \tag{2.3}$$

The CF of the sampled process $\{x(t_k)\}$ is given by

$$\begin{aligned}C(n)&=E[(x(t_i)-\mu)(x(t_j)-\mu)]\\ &=E[E[(x(t_i)-\mu)(x(t_j)-\mu);\quad \text{for fixed } t_i \text{ and } t_j]]\end{aligned} \tag{2.4}$$

where $\mu=E[x]$, $\sigma^2=E[(x-\mu)^2]$ and $n=|i-j|$; the number of intervals between the samples $x(t_i)$ and $x(t_j)$. $E[\cdot]$ denotes expectation. In (2.4) the two expectations are with respect to x and the exponentially distributed random variable $|t_i-t_j|$.

Consider a process $x(t)$ with an exponentially decaying CF $\sigma^2\exp(-\alpha|\tau|)$, where σ^2 is the variance of $x(t)$ and $\alpha>0$. The CF of the sequence $\{x(t_k)\}$ can now be found using (2.4):

$$C(n)=E[\sigma^2\exp(-\alpha|\tau|)]\qquad \tau\geqslant 0 \tag{2.5}$$

where $|\tau|=|t_i-t_j|$ is the random variable. Using (2.3) and (2.5),

$$C(n)=\sigma^2\int_0^\infty \exp(-\alpha\tau)\frac{\nu^n\tau^{n-1}\exp(-\nu\tau)}{(n-1)!}\,\mathrm{d}\tau. \tag{2.6}$$

Then

$$C(n)=\sigma^2\left(\frac{\nu}{\alpha+\nu}\right)^n, \tag{2.7}$$

since the gamma function is defined as

$$\Gamma(n+1)=\int_0^\infty \phi^n\exp(-\phi)\,\mathrm{d}\phi=n!\qquad (n \text{ is natural}).$$

By considering the exponentially decaying CF as $\mathrm{Re}\{\exp[-(\alpha+\mathrm{i}\omega)|\tau|]\}$, $(\mathrm{i}=\sqrt{-1})$, the CF of the sampled sequence is simply

$$C(n)=\mathrm{Re}\left(\frac{\nu}{\nu+(\alpha+\mathrm{i}\omega)}\right)^n \tag{2.8}$$

where Re denotes real part. For the Gaussian CF $\sigma^2\exp(-\lambda\tau^2)$, the covariance of the sequence now becomes

$$C(n)=\int_0^\infty \sigma^2\exp(-\lambda\tau^2)\frac{\nu^n\tau^{n-1}}{\Gamma(n)}\exp(-\nu\tau)\,\mathrm{d}\tau. \tag{2.9}$$

From Gradshteyn and Rhysik (1966)

$$\int_0^\infty x^{\nu-1}\exp(-\beta x^2-\gamma x)\,\mathrm{d}x=(2\beta)^{-\nu/2}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{(2\beta)}}\right)\qquad \beta,\nu>0. \tag{2.10}$$

Therefore

$$C(n)=\sigma^2\left(\frac{\nu}{\surd(2\lambda)}\right)^n \exp\left(\frac{\nu^2}{8\lambda}\right)D_{-n}\left(\frac{\nu}{\surd(2\lambda)}\right)=\sigma^2 z^n \exp(z^2/4)D_{-n}(z) \tag{2.11}$$

where $D_{-n}(z)$ is a parabolic cylinder function and $z=\nu/\surd(2\lambda)$.

3. Covariance function $\sigma^2 \exp(-\alpha|\tau|)$

3.1. Mean estimate and its variance

We shall now consider the mean estimate $\hat{x}$ of the sequence of variables $\{x(t_i)\}$, $i=1,\ldots,n$. Representing $x(t_i)$ by x_i, a reasonable estimate is

$$\hat{x}=\sum_{i=1}^{n} x_i/n. \tag{3.1}$$

Unless specified, all summations that follow will be from 1 to n. Since we know *a priori* that $E[x(t)]=\mu$, then,

$$E[\hat{x}]=E\left[\sum_i x_i/n\right]=\sum_i E[x_i]/n=\mu. \tag{3.2}$$

The variance of the unbiased mean estimator $\hat{x}$ is

$$V(\hat{x})=E[\hat{x}-\mu]^2=\frac{1}{n^2}E\left[\sum_i (x_i-\mu)\right]^2. \tag{3.3}$$

In appendix 1 it is shown that $V(\hat{x})$ can be evaluated as

$$V(\hat{x})=\frac{\sigma^2}{n^2}\left(n+\frac{2a}{(1-a)^2}[n(1-a)+a^n-1]\right) \tag{3.4}$$

where $a=\nu/(\nu+\alpha)$. Since $a<1$, we can deduce that $\hat{x}$ is a consistent estimator since $V(\hat{x})\to 0$ as $n\to\infty$. It should also be noticed that when the samples $\{x(t_k)\}$ are independent, i.e. when $a=0$, then $V(\hat{x})=\sigma^2/n$. This is the standard result for the variance of the mean of n independent samples. For large n,

$$\frac{V(\hat{x})}{\sigma^2/n}\simeq\frac{1+a}{1-a}. \tag{3.5}$$

Figure 1 shows $V(\hat{x})/\sigma^2$ plotted against n for various values of a. The full curves represent the values obtained from (3.4) whilst the broken curves indicate the approximate values given by (3.5). The curves show the range of n for which (3.5) is valid; the full and broken curves being asymptotic at high n values.

3.2. Variance estimator and its variance

Now consider an unbiased variance estimator

$$\hat{v}=\frac{1}{N}\sum_i (x_i-\hat{x})^2 \tag{3.6}$$

where N has to be determined such that $E(\hat{v})=\sigma^2$.

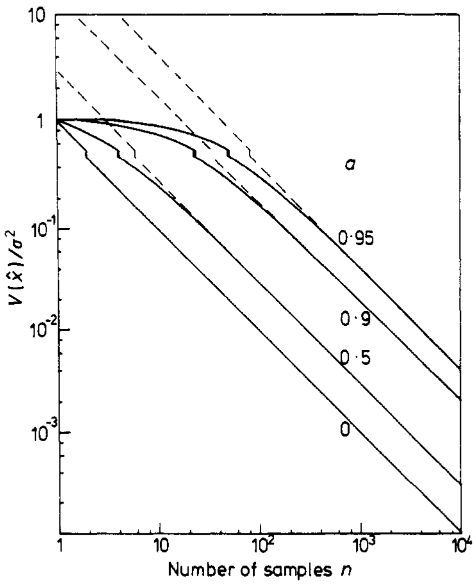

Figure 1. Variation of the mean value estimator with the number of samples for covariance function $\sigma^2 \exp(-\alpha|\tau|)$. The full curves are plotted from (3.4) and the broken curves from (3.5).

In appendix 2 it is shown that the condition for an unbiased estimation is that

$$N = n - 1 - \frac{2a}{n(1-a)^2}[n(1-a) + a^n - 1] \tag{3.7}$$

and that the variance of this estimator is

$$V(\hat{v}) = E[(\hat{v} - \sigma^2)^2] = E(\hat{v}^2) - \sigma^4 \tag{3.8a}$$

$$= \frac{2}{N^2}(I_1 + I_2 + I_3) \tag{3.8b}$$

where

$$I_1 = \sigma^4\left(n + \frac{2a^2}{(1-a^2)^2}[n(1-a^2) + a^{2n} - 1]\right),$$

$$I_2 = \frac{\sigma^4}{n^2}\left(n + \frac{2a}{(1-a)^2}[n(1-a) + a^n - 1]\right)^2$$

and

$$I_3 = -\frac{2\sigma^4}{n(n-a)^2}\left(n(1 + 2a + a^2 + 2a^{n+1}) - 4\frac{(1+a)a(1-a^2)}{(1-a)} + 2\frac{a^2(1-a^{2n})}{(1-a^2)}\right).$$

When $a = 0$, $V(\hat{v}) = 2\sigma^4/(n-1)$ which is the same as given by Hald (1952) for independent samples. It will be noticed that only I_1 contributes significantly to $V(\hat{v})$ for large n. A similar result to (3.5) can be obtained in this case for large n:

$$\frac{V(\hat{v})}{2\sigma^4/(n-1)} \simeq \frac{1+a^2}{1-a^2}. \tag{3.9}$$

This equation can be verified by considering $V(\hat{v})$ in figure 2 for $n = 10^4$ with $a = 0$ and $a = 0{\cdot}9$. From the curves

$$\frac{V(\hat{v}, a = 0{\cdot}9)}{V(\hat{v}, a = 0)} = \frac{1{\cdot}9 \times 10^{-3}}{2 \times 10^{-4}} \simeq 10$$

$$\frac{1 + (0{\cdot}9)^2}{1 - (0{\cdot}9)^2} = \frac{1{\cdot}81}{0{\cdot}19} \simeq 10 \qquad \text{(using (3.9)).}$$

The full curves in figure 2 have been computed from (3.8) and the broken curves from (3.9).

The variance estimator is consistent since $V(\hat{v})$ approaches zero as n increases (see figure 2 and (3.9)).

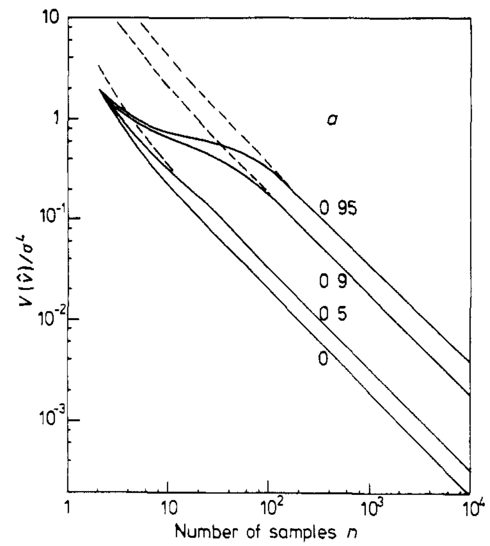

Figure 2. Variation of the variance estimator with the number of samples for covariance function $\sigma^2 \exp(-\alpha|\tau|)$. The full curves are plotted from (3.8) and the broken curves from (3.9).

4. Covariance function $\sigma^2 \exp(-\alpha|\tau|) \cos \omega\tau$

As indicated in § 2, all the results obtained in § 3 can be directly extended to the present CF.

For example, using (3.4) the variance of the mean estimator is

$$V(\hat{x}) = \mathrm{Re}\left[\frac{\sigma^2}{n^2}\left(n + \frac{2a}{(1-a)^2}[n(1-a) + a^n - 1]\right)\right] \qquad (4.1)$$

where $a = \nu/[\nu + (\alpha + \mathrm{i}\omega)]$. Similarly the variance of the variance estimator can be obtained.

By setting $\alpha = 0$, a periodic covariance function can be considered.

5. Covariance function $\sigma^2 \exp(-\lambda\tau^2)$

In this case the variance of the estimators cannot be derived analytically; however, the required results can easily be obtained numerically.

5.1. Variance of the mean estimator

By comparing (2.11) with (2.7) we can see that $a^n \equiv z^n \exp(z^2/4)D_{-n}(z)$. Using this similarity and the results of § 3.1 and appendix 1 the variance of the mean estimate is seen to be

$$V(\hat{x}) = \frac{1}{n^2}\Big(n\sigma^2 + 2\sigma^2 \sum_{i>j}\sum z^{i-j} \exp(z^2/4) D_{-(i-j)}(z)\Big)$$

$$= \frac{\sigma^2}{n^2}\Big(n + 2\exp(z^2/4)\sum_r (n-r) z^r D_{-r}(z)\Big). \tag{5.1}$$

The summation in (5.1) is convergent and the value of $z^r D_{-r}(z)$ can be computed using the following recurrence relation:

$$z^{r+2} D_{-(r+2)}(z) = \frac{z^2}{r+1}\Big(z^r D_{-r}(z) - z^{r+1} D_{-(r+1)}(z)\Big). \tag{5.2}$$

Care should be taken when using this recurrence formula because of the round-off errors in computation. A backward recurrence technique described in detail by Abramowitz and Stegun (1965) is used to avoid such errors. For values of r beyond a certain value k (dependent on z) $z^k D_{-k}(z)$ is approximately zero. We let $z^k D_{-k}(z)$ and $z^{k+1} D_{-(k+1)}(z)$ be 1 and 0 respectively. Backward recurrence is then used until the value of $zD_{-1}(z)$ is obtained, which is then compared with the tabulated values in Abramowitz and Stegun (1965). The ratio of the tabulated to the computed value for $r = 1$ is obtained. Multiplication of all the computed values by this ratio gives the correct value for all $r \leqslant k+1$.

The full curves in figure 3 are for $V(\hat{x})/\sigma^2$ against n for various values of z computed from (5.1). It should be noticed that the $z = 0$ curve corresponds to the independent-samples case and is the same as that obtained for $a = 0$ when considering the exponentially decaying correlation (see figure 1).

For large n, $V(\hat{x})/\sigma^2$ decreases linearly with n. The variance of the mean estimate as n approaches infinity can be obtained analytically. Rewriting (5.1)

$$V(\hat{x}) = \frac{\sigma^2}{n^2}\Big(n + 2\exp(z^2/4)\sum_r (n-r) z^r D_{-r}(z)\Big). \tag{5.3}$$

Using (2.10) we obtain

$$\sum_r z^r D_{-r}(z) = \exp(-z^2/4) \sum_r z \int_0^\infty \frac{(xz)^{r-1}}{(r-1)!} \exp\Big(-zx - \frac{x^2}{2}\Big)\,dx$$

$$= z\exp(-z^2/4) \int_0^\infty \sum_r \frac{(xz)^{r-1}}{(r-1)!} \exp\Big(-zx - \frac{x^2}{2}\Big)\,dx \tag{5.4}$$

and as n, the upper limit of the summation, approaches infinity

$$\sum_r (xz)^{r-1}/(r-1)! \simeq e^{xz}.$$

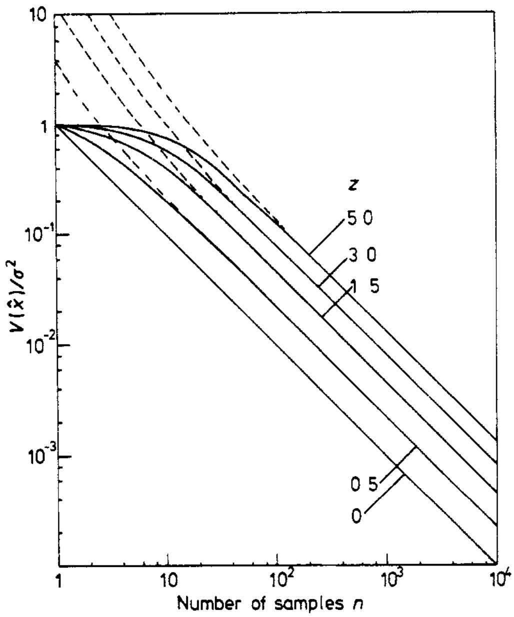

Figure 3. Variation of the mean value estimator with the number of samples for covariance function $\sigma^2 \exp(-\lambda\tau^2)$. The full curves are plotted from (5.1) and the broken curves from (5.7*a*).

Therefore

$$\sum_r z^r D_{-r}(z) = z \exp(-z^2/4) \int_0^\infty \exp(-x^2/2)\,\mathrm{d}x = z \exp(-z^2/4)(\pi/2)^{1/2}. \qquad (5.5)$$

Interchanging the order of summation and integration

$$\sum_r r z^r D_{-r}(z) = z \exp(-z^2/4) \int_0^\infty \sum_r r \frac{(zx)^{r-1}}{(r-1)!} \exp\left(-zx - \frac{x^2}{2}\right) \mathrm{d}x.$$

From Gradshteyn and Rhysik (1966)

$$\sum_r \frac{r(zx)^{r-1}}{(r-1)!} \simeq zx \exp(zx) + \exp(zx) \qquad \text{as } n \to \infty. \qquad (5.6)$$

Substituting (5.5) and (5.6) in (5.3) we find

$$V(\hat{x}) = \frac{\sigma^2}{n}\left(1 + z(2\pi)^{1/2} + \frac{1}{n}[2z^2 + z(2\pi)^{1/2}]\right) \qquad (5.7a)$$

and after neglecting the $1/n^2$ term

$$V(\hat{x}) = \frac{\sigma^2}{n}[1 + z(2\pi)^{1/2}]. \qquad (5.7b)$$

This relationship can be verified easily by considering the curves in figure 3. The broken curves in figure 3 have been computed from (5.7*a*).

5.2. Variance of the variance estimator

The results of § 3.2 can be simply extended to this covariance function. From (3.8*b*)

$$V(\hat{v}) = \frac{2}{N^2}(I_1 + I_2 + I_3) \tag{5.8a}$$

where

$$I_1 = \sigma^4\Big(n + 2\exp(z^2/2)\sum_r (n-r)(z^r D_{-r}(z))^2\Big) \tag{5.8b}$$

$$I_2 = \Big(\sum_i A(i)\Big)^2 \tag{5.8c}$$

$$I_3 = n\sum_i A^2(i) \tag{5.8d}$$

$$A(i) = \frac{\sigma^2}{n}\Big[1 + \exp(z^2/4)\Big(\sum_{r=1}^{n-i} z^r D_{-r}(z) + \sum_{r=1}^{i-1} z^r D_{-r}(z)\Big)\Big] \tag{5.8e}$$

and

$$N = n - 1 - (2/n)\exp(z^2/4)\sum_r (n-r) z^r D_{-r}(z). \tag{5.8f}$$

As in § 3.2, only I_1 contributes significantly to $V(\hat{v})$ for large n. Hence,

$$V(\hat{v}) \simeq \frac{2\sigma^4}{(n-1)^2}\Big(n + 2n\exp(z^2/2)\sum_r (z^r D_{-r}(z))^2\Big). \tag{5.9}$$

$\Sigma_r\, (z^r D_{-r}(z))^2$ converges very quickly to zero when $z < 1$, hence it is justifiable to retain only two terms of the summation, therefore

$$V(\hat{v}) \simeq \frac{2\sigma^4 n}{(n-1)^2}[1 + 2\exp(z^2/2)(z^2 D_{-1}^2(z) + z^4 D_{-2}^2(z))]. \tag{5.10}$$

Figure 4 shows graphs of $V(\hat{v})/\sigma^4$ against n for different values of z. The full curves have been computed from (5.8). For $z = 0{\cdot}5$ the broken curve has been computed from (5.10) whilst for $z = 1, 2, 3$ and 5 they have been evaluated from (5.9).

6. Continuous averaging

The results obtainable by continuous averaging (hot-wire anemometry) will be compared with those obtained by averaging a Poisson-sampled signal (laser anemometry). The latter has been discussed in the previous sections. The most commonly used model for the CF of the velocity is $\sigma^2\exp(-\alpha|\tau|)$ and hence we shall only consider this one.

The estimates of mean and variance respectively for continuous averaging are:

$$\hat{x}_c = \frac{1}{T}\int_0^T x(t)\,dt$$

$$\hat{v}_c = \frac{1}{P}\int_0^T \Big(x(t) - \frac{1}{T}\int_0^T x(t)\,dt\Big)^2\,dt$$

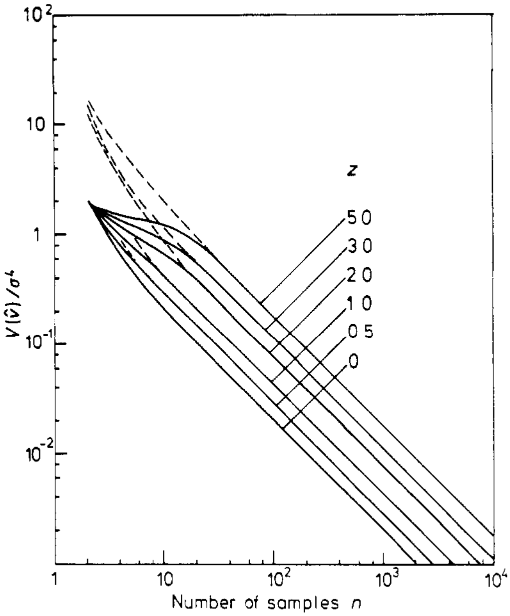

Figure 4. Variation of the variance estimator with the number of samples for covariance function $\sigma^2 \exp(-\lambda\tau^2)$. The full curves are plotted from (5.8). The broken curves are plotted from (5.10) when $z = 0{\cdot}5$ and from (5.9) when $z = 1, 2, 3$ and 5.

where $x(t)$ is the continuous velocity record with length T. P is such that the variance estimator is unbiased.

Using a procedure similar to that given by Bendat and Piersol (1971) we find the variance of these estimators for large T:

$$V(\hat{x}_c) = \frac{2}{T}\int_0^\infty C(\tau)\,d\tau = \frac{2}{T}\int_0^\infty \sigma^2 \exp(-\alpha\tau)\,d\tau = \frac{2\sigma^2}{T\alpha}$$
$$V(\hat{v}_c) = \frac{2T}{P^2}\left(2\int_0^\infty C^2(\tau)\,d\tau\right) = \frac{4T}{P^2}\int_0^\infty \sigma^4 \exp(-2\alpha\tau)\,d\tau = \frac{2\sigma^4 T}{P^2\alpha}. \qquad (6.1)$$

It can be shown that $P = T - (2/\alpha)$, hence

$$V(\hat{v}_c) = 2\sigma^4/\alpha T. \qquad (6.2)$$

Although the variances of the estimators for the sampled process are higher than those for continuous averaging, it will be shown that the results of the former approach those of the latter as the rate of the Poisson process, $\nu = n/T$, approaches infinity.

Rewriting (3.5) and using the definition of 'a'

$$V(\hat{x}) = \frac{\sigma^2}{n}\frac{1+a}{1-a} = \frac{\sigma^2}{n}\left[\left(\frac{(\alpha/\nu)+1+1}{(\alpha/\nu)+1-1}\right)\right] \simeq \frac{\sigma^2}{n}\left(\frac{2}{\alpha/\nu}\right) = \frac{2\sigma^2}{\alpha T}. \qquad (6.3)$$

From (3.9)

$$V(\hat{v}) = \frac{2\sigma^4}{n}\frac{1+a^2}{1-a^2} = \frac{2\sigma^4}{\alpha T}. \qquad (6.4)$$

Appendix 1. Variance of the mean estimator for CF $\sigma^2 \exp(-\alpha|\tau|)$

The variance of the mean estimator $\hat{x}$ is

$$V(\hat{x}) = E[\hat{x}-\mu]^2 = \frac{1}{n^2} E\Big[\sum_i (x_i-\mu)\Big]^2$$

$$= \frac{1}{n^2} E\Big[\sum_i (x_i-\mu)^2 + \sum_{i\neq j}\sum (x_i-\mu)(x_j-\mu)\Big]$$

$$= \frac{1}{n^2}\Big(\sum_i E[(x_i-\mu)^2] + \sum_{i\neq j}\sum E[(x_i-\mu)(x_j-\mu)]\Big). \tag{A.1}$$

By definition of variance

$$E[(x_i-\mu)^2] = C(0) = \sigma^2. \tag{A.2}$$

Using (2.7) and (A.2)

$$V(\hat{x}) = \frac{1}{n^2}\Big(n\sigma^2 + \sigma^2 \sum_{i\neq j}\sum \Big(\frac{\nu}{\nu+\alpha}\Big)^{|i-j|}\Big). \tag{A.3}$$

Letting $a = \nu/(\nu+\alpha) < 1$ and removing the modulus sign we obtain

$$\sum_{i\neq j}\sum \Big(\frac{\nu}{\nu+\alpha}\Big)^{|i-j|} = 2\sum_{i>j}\sum a^{i-j} \tag{A.4a}$$

$$= 2\sum_r (n-r)a^r. \tag{A.4b}$$

In expanding (A.4a) we find exactly $2(n-r)$ terms of a^r where $r = 1, 2, \ldots, n-1$, therefore (A.3) becomes

$$V(\hat{x}) = \frac{1}{n^2}\Big(n\sigma^2 + 2\sigma^2 \sum_r (n-r)a^r\Big). \tag{A.5}$$

Equation (3.4) then follows using the following results from Gradshteyn and Rhysik (1966):

$$\sum_r a^r = \frac{a(1-a^n)}{1-a} \tag{A.6}$$

$$\sum_r ra^r = a\Big(\frac{1-(n+1)a^n + na^{n+1}}{(1-a)^2}\Big). \tag{A.7}$$

Appendix 2. Variance estimator and its variance for CF $\sigma^2 \exp(-\alpha|\tau|)$

The expected value of the variance estimator is

$$E[\hat{v}] = \frac{1}{N}\sum_i E[(x_i-\hat{x})^2] \tag{A.8}$$

$$E[(x_i-\hat{x})^2] = E[\{(x_i-\mu)-(\hat{x}-\mu)\}^2] = E[(x_i-\mu)^2] + E[(\hat{x}-\mu)^2] - 2E[(x_i-\mu)(\hat{x}-\mu)]. \tag{A.9}$$

Using (3.3) we obtain from (A.8) and (A.9)

$$E[\hat{v}]=\frac{1}{N}\sum_i\left(\sigma^2+V(\hat{x})-2E\left[(x_i-\mu)\frac{1}{n}\sum_j(x_j-\mu)\right]\right)$$

$$=\frac{1}{N}\left(n\sigma^2+nV(\hat{x})-\frac{2}{n}\sum_i\sum_j\sigma^2a^{|i-j|}\right)$$

$$=\frac{1}{N}\left(n\sigma^2+nV(\hat{x})-2\sigma^2-\frac{2\sigma^2}{n}\sum_{i\neq j}\sum a^{|i-j|}\right). \tag{A.10}$$

Employing (A.4) and (A.5) we obtain

$$E[\hat{v}]=\frac{1}{N}\left(n\sigma^2+nV(\hat{x})-2\sigma^2-\frac{4\sigma^2}{n}\sum_r(n-r)a^r\right)$$

$$=\frac{1}{N}\left(n\sigma^2-\sigma^2-\frac{2\sigma^2}{n}\sum_r(n-r)a^r\right). \tag{A.11}$$

For $\hat{v}$ to be unbiased $E[\hat{v}]=\sigma^2$, hence

$$N=n-1-\frac{2}{n}\sum_r(n-r)a^r=n-1-\frac{2a}{n(1-a)^2}[n(1-a)+a^n-1]. \tag{A.12}$$

Using (A.5), N can be expressed as a function of $V(\hat{x})$, i.e.

$$N=n-nV(\hat{x})/\sigma^2=n-V(\hat{x})/(\sigma^2/n). \tag{A.13}$$

When the samples are independent it can be shown that $N=n-1$, by putting $a=0$ in (A.11) or substituting $V(\hat{x})=\sigma^2/n$ in (A.13).

The variance of the variance estimator may be written as

$$V(\hat{v})=E[(\hat{v}-\sigma^2)^2]=E[\hat{v}^2]-\sigma^4.$$

This can be simplified as follows:

$$E[\hat{v}^2]=E\left[\left(\frac{1}{N}\sum_i(x_i-\hat{x})^2\right)^2\right]=\frac{1}{N^2}\sum_i\sum_j E[(x_i-\hat{x})^2(x_j-\hat{x})^2]. \tag{A.14}$$

Assuming Gaussian statistics for x_i and letting $z_i=x_i-\hat{x}$, then z_i has zero mean and

$$E[z_i^2z_j^2]=E[z_i^2]E[z_j^2]+2E^2[z_iz_j]. \tag{A.15}$$

It can be shown that $\Sigma_i E[z_i^2]=N\sigma^2$. Also,

$$E[z_iz_j]=E[(x_i-\mu)(x_j-\mu)]$$
$$+E[(\hat{x}-\mu)(\hat{x}-\mu)]-E[(x_i-\mu)(\hat{x}-\mu)]-E[(x_j-\mu)(\hat{x}-\mu)]. \tag{A.16}$$

Let

$$A(i)=E[(x_i-\mu)(\hat{x}-\mu)]=\frac{1}{n}\sum_j E[(x_i-\mu)(x_j-\mu)]$$

$$=\frac{1}{n}\sum_j\sigma^2a^{|i-j|}=\frac{\sigma^2}{n}\left(n+\sum_{r=1}^{n-i}a^r+\sum_{r=1}^{i-1}a^r\right). \tag{A.17}$$

Then

$$\sum_i A(i)=nV(\hat{x}). \tag{A.18}$$

Substituting (A.17) in (A.16) gives

$$E[z_i z_j] = \sigma^2 a^{|i-j|} + V(\hat{x}) - A(i) - A(j). \tag{A.19}$$

Therefore

$$\sum_i \sum_j E^2[z_i z_j] = \sum_i \sum_j (\sigma^2 a^{|i-j|} + V(\hat{x}) - A(i) - A(j))^2. \tag{A.20}$$

Using (A.17) and simplifying:

$$\sum_i \sum_j E^2[z_i z_j] = I_1 + I_2 + I_3 \tag{A.21a}$$

where

$$I_1 = \sum_i \sum_j \sigma^4 a^{2|i-j|}, \qquad I_2 = n^2 V^2(\hat{x}), \qquad I_3 = -2n \sum_i A_i^2. \tag{A.21b}$$

Substituting (A.14), (A.15) and (A.21) into (3.8*a*) and employing (A.4), (3.4) and (A.6) then leads to the required result (3.8*b*).

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